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NOTE ON NORMAL SECTIONS OF A SURFACE IN A SPACE OF N DIMENSIONS.

By C. L. E. Moore.

1. In his dissertation Karl Kommerell* discussed surfaces (two-dimensional configurations of points) in a space of four dimensions and generalized many of the well-known properties of surfaces in ordinary space. The method used was that of ordinary theory of surfaces. In this note some of these properties are generalized for surfaces in a space of n dimensions. For the properties considered the generalization is complete when $n = 5$. The method here used is that of vector analysis and the notation used is that of Grassmann's Ausdehnungslehre.†

2. If we represent a surface vectorially by the equation

$$x = x(u, v)$$

then the unit tangent vector and curvature vector of a curve traced on the surface are dx/ds and d^2x/ds^2, where s denotes the length of arc of the curve. The differentials dx and d^2x will denote vectors directed respectively along the tangent and the principal normal. The osculating plane of a curve on the surface is defined by

$$[dxd^2x], \tag{1}$$

which on expanding becomes

$$\begin{aligned}&[(x_1du + x_2dv)(x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + x_1d^2u + x_2d^2v]\\ &\qquad = [(x_1du + x_2dv)X] + [x_1x_2](dud^2v - dvd^2u),\end{aligned} \tag{2}$$

where

$$x_1 = \frac{\partial x}{\partial u}, \qquad x_2 = \frac{\partial x}{\partial v}, \qquad x_{11} = \frac{\partial^2 x}{\partial u^2}, \qquad \text{etc.},$$

and

$$X = x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2.$$

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* "Die Krümmung der zwei dimensionalen Gebilde im ebenen Raum von vier Dimensionen," Tübingen, 1897. A synopsis of the results of this paper is found in an article by the same author, "Riemannsche Flächen im ebenen Raum von vier Dimensionen." Mathematische Annalen, vol. 60, 1905.

† Gesammelte Werke, vol. I. Good expositions of this analysis can also be found in the following places: Encyclopédie, édition Français, Vol. I.; Nombres Complexes, Study-Carton; Universal Algebra, Whitehead; Application de la methode vectorielle, Fehr; Louis Ingold, Transactions of the American Mathematical Society, vol. 11 (1910).

If du/dv is held fixed and d^2u/d^2v is allowed to vary, since the parameter $(dud^2v - dvd^2u)$ enters linearly, we see that the plane (2) generates a space of three dimensions. This S_3 is given by

$$[x_1 x_2 X]. \tag{3}$$

For values of d^2u/d^2v can be found so that the product of x_1, x_2 or X with the plane (2) will vanish and therefore the S_3 must be that determined by these three vectors. The tangent plane to the surface is $[x_1x_2]$ and is therefore contained in the S_3 above. This is called *the osculating 3-space* and is said to osculate the surface along the direction du/dv. If du/dv also varies, the osculating S_3's will generate a quadric cone of four dimensions, V_4^2, since X is quadratic in du/dv. From equations (2) and (3) we see that this cone lies in the space of five dimensions

$$[x_1 x_2 x_{11} x_{12} x_{22}], \tag{4}$$

and has the tangent plane for vertex.*

3. The normals to a surface at a point P, that is lines perpendicular to the tangent plane, will generate a linear space of $n - 2$ dimensions, which we will denote by N_{n-2}. The normal space is the complement of the tangent plane $[x_1x_2]$ and according to Grassmann will be denoted by $|[x_1x_2]$. The osculating S_3 along a given direction du/dv will cut N_{n-2} in a line normal to the surface. Hence, *of all the curves traced on a surface which are tangent to a fixed direction at a fixed point there is just one whose osculating plane contains a normal to the surface.*

Let us confine our attention, for the moment, to a given osculating S_3. Call the *unit* normal lying in it n and the unit principal normal to the curve ν. The angle between these two directions is given by the formula

$$[n|\nu] = \cos\theta.$$

From the Frenet formulæ we have

$$\frac{d^2x}{ds^2} = \frac{\nu}{\rho},$$

where ρ denotes the radius of curvature of the curve, and consequently

$$\left[\frac{d^2x}{ds^2}\middle| n\right] = \frac{1}{\rho}[\nu|n] = \frac{\cos\theta}{\rho}.$$

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* These results are due to Del Pezzo, "Sugli spazi tangenti ad una superficie o ad una varietá immersa in uno spazio di piu dimensione." Rendiconti Accademia della Scienze di Napoli, 1886. The treatment is entirely synthetic. Segre obtained the same results analytically in a paper, "Su una classa di superficie degl' iperspazi, ecc.," Atti di Torino, 1907.

Expanding the right side of this equation, we have

$$[(x_{11}u'^2 + 2x_{12}u'v' + x_{22}v'^2 + x_1u'' + x_2v'')|n] =$$

$$[(x_{11}u'^2 + 2x_{12}u'v' + x_{22}v'^2)|n] = \frac{\cos\theta}{\rho}.$$

(The accents denote derivatives with respect to the arc length.) The right side of this equality does not depend on the second order derivatives of u and v. Therefore if du/dv is held fixed, that is, if we consider a fixed osculating S_3, we have

$$\frac{\cos\theta}{\rho} = \text{const.} = \frac{1}{R},$$

or

$$\rho = R\cos\theta. \tag{5}$$

Here R denotes the radius of curvature of the curve whose osculating plane contains the normal n of the surface, that is the radius of curvature of the normal section of the surface corresponding to the direction du/dv. Meusnier's theorem for surfaces in space of n dimensions is then: *On a surface in S_n the osculating circles to curves passing through a fixed point and tangent to a fixed direction generate a sphere tangent to the surface.*

These results are readily generalized for any variety V_m in S_n. The osculating planes of curves tangent to a fixed direction generate an S_{m+1} which contains the tangent S_m. There is just one of these curves whose principal normal is normal to the variety. In this S_{m+1} Meusnier's theorem is the same as for a V_m in S_{m+1}.

4. Corresponding to each direction on the surface there is one curve whose principal normal is normal to the surface. This one parameter family of normals, corresponding to the different directions traced on the surface, will generate a quadric cone. For the osculating S_3's generate a cone V_4^2. This cone is cut by the normal space N_{n-2} in a quadric cone lying in the S_3 formed by the intersection of N_{n-2} and the S_5, $[x_1x_2x_{11}x_{12}x_{22}]$. Hence, *the curvature vectors of curves traced on a surface through a fixed point and whose osculating planes are normal to the surface, generate a quadric cone lying in an S_3.* We will call these curves normal sections of the surface.

From this it is seen that so far as curvature is concerned we have the complete generalization when we consider surfaces in S_5. This would follow from the fact also that the points of a surface near a fixed point to infinitesimals of second order all lie in a space of five dimensions.

If the cone V_4^2 degenerates, that is if

$$[x_1x_2x_{11}x_{12}x_{22}] = 0, \tag{6}$$

the osculating S_3's generate an S_4, and the curvature vectors of the normal sections will generate a plane pencil lying in the plane of intersection of this S_4 with N_{n-2}.

Segre* has developed the projective theory of the surfaces whose coördinates satisfy relation (6) at each point. So far as curvature is concerned it is evident that these surfaces have a very close resemblance to surfaces in a space of four dimensions. Kommerell showed that in S_4 an osculating S_3 (or in this case any hyperplane passed through the tangent plane) osculates the surface in two different directions. We will now show that this is also true for surfaces in S_n which satisfy relation (6). The osculating S_3 is

$$[x_1x_2(x_{11}\tau^2 + 2x_{12}\tau + x_{22})] \qquad \tau = du/dv,$$

and (6) is equivalent to

$$(C) \qquad Ax_{11} + 2Bx_{12} + Cx_{22} + Dx_1 + Ex_2 = 0.$$

Eliminating x_{22}, for example, we have

$$(3') \qquad [x_1x_2x_{11}]\left(\tau^2 - \frac{A}{c}\right) + [x_1x_2x_{12}]\left(\tau - \frac{B}{c}\right) \qquad \text{(for } \tau \text{ fixed).}$$

Any osculating S_3 can be put into the form

$$[x_1x_2x_{11}]\left(t^2 - \frac{A}{c}\right) + [x_1x_2x_{12}]\left(t - \frac{B}{c}\right) \qquad \text{(for } t \text{ variable).}$$

The second will coincide with the first, if

$$(7) \qquad \frac{Ct^2 - A}{C\tau^2 - A} = \frac{Ct - B}{C\tau - B}.$$

Since this relation is quadratic in t there are two directions along which it will osculate the surface. It is easily verified that these two directions will coincide if

$$(8) \qquad C\tau^2 - 2B\tau + A = 0.$$

This is the equation which defines the characteristics of the equation (C). Equation (3′) shows that any S_3 in the S_4 defined by $[x_1x_2x_{11}x_{12}x_{22}] = 0$ and containing the tangent plane is an osculating S_3. If then we write any osculating S_3 in the form

$$[x_1x_2x_{11}]M + [x_1x_2x_{12}]N,$$

the directions along which it osculates the surface are given by

$$\frac{Ct^2 - A}{M} = \frac{Ct - B}{N}.$$

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* Loc. cit.

It is easily seen that these two directions are harmonic with respect to the two directions defined by (8). Two directions along which the same S_3 osculates are called conjugate directions. *The directions of the characteristics are self-conjugate and are separated harmonically by each pair of conjugate directions.*

If not only (6) but also equations

$$(9) \qquad [x_1x_2x_{11}x_{12}] = 0, \qquad [x_1x_2x_{11}x_{22}] = 0$$

are satisfied, the same S_3 will osculate in all directions. For in this case the five vectors x_1, x_2, x_{11}, x_{12}, x_{22} lie in an S_3. The normal is the intersection of this S_3 and N_{n-2}. All the normal sections of the surface determine the same normal to the surface. Segre* has shown that if two such equations are satisfied, and n is greater than three the surface must be a developable, that is formed by the tangent lines to a twisted curve. These surfaces then, so far as curvature is concerned, behave like surfaces in ordinary space.

5. If we make u and v functions of a single variable t, the magnitude of the curvature of the curve thus obtained is

$$(11) \qquad c = \frac{\sqrt{[x''x']|[x''x']}}{[x'|x']^{\frac{3}{2}}},$$

accents denoting differentiation with respect to t. For let τ denote the unit tangent, then

$$\frac{dx}{ds} = \tau = \frac{dx}{dt}\frac{dt}{ds} = \frac{x'}{s'}, \qquad s' = \frac{ds}{dt},$$

Then

$$x' = s'\tau, \qquad x'' = s''\tau + s'\tau'.$$

But

$$[x'x''] = [s'\tau(s''\tau + s'\tau)] = s'^2[\tau\tau'].$$

$$c = \frac{d\tau}{ds} = \frac{\tau'}{s'},$$

or

$$\tau' = s'\mathbf{c}.$$

$\mathbf{c}$ denotes the curvature vector. Substituting this value, we have

$$[x'x''] = s'^2[x'\mathbf{c}].$$

Taking the inner product of this with itself, we have

$$[x'x'']|[x'x''] = s'^4[x'\mathbf{c}]|[x'\mathbf{c}] = s'^6\mathbf{c}^2,$$

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* Loc. cit.

or

$$c = \frac{\sqrt{[x_1x'']|[x'x'']}}{s'^3}.$$

But $s'^2 = [x'|x']$, which substituted gives formula (11). If we take $z_1, z_2, \cdots z_n$ as the components of the vector x we have

$$c = \frac{1}{\rho} = \frac{\sqrt{\Sigma(z_i'z_j'' - z_j'z_i'')^2}}{\{\Sigma z_i'^2\}^{\frac{3}{2}}}.$$

If we take the origin on the surface and take the z_1, z_2 axes tangent to the parameter curves, then the components of the vector x can be written in the form

$$\begin{aligned} z_1 &= u, \qquad z_2 = v, \\ z_3 &= a_3u^2 + 2b_3uv + c_3v^2 + \cdots, \\ &\cdots\cdots\cdots\cdots \\ z_n &= a_nu^2 + 2b_nuv + c_nv^2 + \cdots. \end{aligned} \tag{12}$$

A curve on the surface can now be defined by

$$v = \lambda u. \tag{13}$$

This is, in fact, a curve cut from the surface by a hyperplane containing the normal N_{n-2}. For such a curve we then have at the origin

$$z_1 = z_2 = z_3 = \cdots z_n = 0. \quad \frac{dv}{du} = \lambda, \quad \frac{d^2v}{du^2} = 0.$$

Using these values the formula for the curvature becomes

$$c^2 = \Sigma\left(\frac{a_i + 2b_i\lambda + c_i\lambda^2}{1 + \lambda^2}\right)^2, \qquad (i = 3, 4, \cdots n). \tag{14}$$

Putting in the condition for a maximum or minimum, namely

$$\frac{dc^2}{d\lambda} = 0,$$

we have

$$\Sigma(a_i + 2b_i\lambda + c_i\lambda^2)(a_i\lambda^2 + (a_i - c_i)\lambda - b_i) = 0.$$

Since this equation is of the fourth degree in λ there are four directions of maximum or minimum curvature.

Kommerell* pointed out that the asymptotic lines on an ordinary surface in 3-space satisfy this condition. The theorem that through each point on a surface in any number of dimensions pass four lines of curvature, is general if we admit the asymptotic lines in ordinary space. The right side of (14) in the case of ordinary space has a single term

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* Loc. cit.

$$c = \frac{a_3 + 2b_3\lambda + c_3\lambda^2}{1 + \lambda^2},$$

and in this case there are real values of λ which will make c zero if $b_3^2 - a_3c_3 > 0$. But for $n > 3$ we see that there are no real values of λ which will make the curvature zero on a real surface.

6. To every direction on the surface corresponds a direction in N_{n-2}. This is the intersection of the osculating S_3 with N_{n-2}. Expressed as a product this is

$$(15) \qquad [x_1x_2X]|[x_1x_2] = [x_1x_2X][z_3z_4 \cdots z_n].$$

The complement of the tangent plane $[x_1x_2]$ is $[z_3z_4 \cdots z_n]$, since x_1, x_2 coincide with z_1, z_2. Developing this product according to elements of the last factor, we have

$$[x_1x_2Xz_4 \cdots z_5]z_3 - [x_1x_2Xz_3z_5 \cdots z_n]z_4 + [x_1x_2Xz_3z_4z_6 \cdots z_n]z_5 + \cdots.$$

The directions of the normal corresponding to the direction du/dv are then proportional to the coefficients of the z's in this vector. From equations (12) and (13) we have

$$[x_1x_2Xz_4z_5 \cdots z_n] = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & A \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & A_4 & \cdots & A_n \\ 0 & 0 & 0 & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = A_3,$$

where $A_i = a_i + 2b_i\lambda + c_i\lambda^2$. Making use of (14), we can put the normal curvature vector in the form

$$c = \frac{1}{1 + \lambda^2}(A_3z_3 + A_4z_4 + \cdots + A_nz_n].$$

The locus of the end of the vector is then,

$$z_i = c \cos \alpha_i,$$

where c is the length of the curvature vector and α_i the angle which it makes with the z_i-axis. The parametric equations of the locus are then,

$$(16) \qquad z_i = \frac{\sqrt{\Sigma A_i^2}}{1 + \lambda^2} \cdot \frac{A_i}{\sqrt{\Sigma A_i^2}} = \frac{A_i}{1 + \lambda^2} = \frac{a_i + 2b_i\lambda + c_i\lambda^2}{1 + \lambda^2}.$$

Therefore, *the locus of the end of the normal curvature vector is a conic.*

The radius of curvature is $1/c$. Hence *the locus of the centers of normal curvature is the inverse of the above conic with respect to a unit circle and is therefore a twisted quartic lying on the quadric cone generated by the normal curvature vectors.*

From equation (14) we saw that c could neither be zero nor infinite for a real direction on the surface. The conic (16) must then be an ellipse and cannot pass through the point on the surface. For a real point on the surface there are two imaginary directions for which the curvature is infinite or the radius of curvature zero. But for an ordinary real point there is no direction for which the curvature is zero. Equation (14) however shows that there are directions such that the magnitude of the curvature is zero. This would be the case if

$$\Sigma A_i^2 = 0.$$

That is if the magnitude of the curvature is zero, the curve has the direction of a minimum line. Equation (14) however shows that the corresponding point of the locus (16) will not be at the origin unless $A_i = 0$, which would not be the case for a generic point of a general surface. If this last relation is satisfied however, there is a direction such that the curvature of a normal section is zero which would mean that this direction has three point contact with the surface.

If the coördinates of the surface satisfy equation (6) the cone of normal radii of curvature will degenerate into a plane but the conic (16) will not degenerate. These surfaces then behave like a surface in S_4.

From the fact that the end of the curvature vector traces a conic can be seen at once that there are four maxima and minima for **c** and hence passing through each point of the surface are four lines of curvature.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.